

QUANTITATIVE UNIQUE CONTINUATION PRINCIPLE FOR SCHRÖDINGER OPERATORS WITH SINGULAR POTENTIALS

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ABSTRACT. We prove a quantitative unique continuation principle for Schrödinger operators $H = -\Delta + V$ on $L^2(\Omega)$, where Ω is an open subset of \mathbb{R}^d and V is a singular potential: $V \in L^\infty(\Omega) + L^p(\Omega)$. As an application, we derive a unique continuation principle for spectral projections of Schrödinger operators with singular potentials.

1. INTRODUCTION

We prove a quantitative unique continuation principle for Schrödinger operators $H = -\Delta + V$ on $L^2(\Omega)$, where Ω is an open subset of \mathbb{R}^d , Δ is the Laplacian operator, and V is a singular real potential: $V \in L^\infty(\Omega) + L^p(\Omega)$. Our results extend the original result of Bourgain and Kenig [BK, Lemma 3.10], as well as subsequent versions [GK2, Theorem A.1] and [BK1, Theorem 3.4], where V is a bounded potential: $V \in L^\infty(\Omega)$.

As an application, we derive a unique continuation principle for spectral projections of Schrödinger operators with singular potentials, extending the bounded potential results of [K1, Theorem 1.1] and [K1N, Theorem B.1].

To prove the quantitative unique continuation principle for singular potentials we use Sobolev inequalities (not required for bounded potentials). Since the Sobolev inequality we use in dimension $d = 2$ is expressed in terms of Orlicz norms, we review Orlicz spaces, following [RR]. A function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ is called a Young function if it is increasing, convex, $\varphi(0) = 0$, and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. Its complementary function, given by $\varphi^*(t) = \sup_{s \in \mathbb{R}^+} \{st - \varphi(s)\}$ for $t \in \mathbb{R}^+$, is also a Young function. Given a Young function φ and a σ -finite measure μ on a measurable space X , we define the Orlicz space

$$(1.1) \quad L^\varphi(X) = \left\{ f : X \rightarrow \mathbb{R} \text{ measurable} \left| \int_X \varphi(\alpha|f|) d\mu < \infty \text{ for some } \alpha > 0 \right. \right\},$$

a Banach space when equipped with the Orlicz norm

$$(1.2) \quad \|f\|_\varphi := \inf \left\{ k > 0 : \int_X \varphi\left(\frac{1}{k}|f|\right) d\mu \leq 1 \right\}.$$

(A standard example is $\varphi(t) = t^p$ with $1 \leq p < \infty$; in this case $L^\varphi(X) = L^p(X)$.) There is a Hölder's inequality for Orlicz spaces:

$$(1.3) \quad \int_X |fg| d\mu \leq 2\|f\|_\varphi \|g\|_{\varphi^*} \quad \text{for all } f \in L^\varphi(X), g \in L^{\varphi^*}(X).$$

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We now state our main theorem, a quantitative unique continuation principle for Schrödinger operators with singular potentials. We fix the Young function

$$(1.4) \quad \varphi(t) = e^t - 1, \quad \text{so} \quad \varphi^*(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ t \log t - t + 1 & \text{if } t > 1 \end{cases}.$$

We use the norm $|x| := (\sum_{j=1}^d |x_j|^2)^{\frac{1}{2}}$ for $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$; all distances in \mathbb{R}^d will be measured with respect to this norm. By $B(x, \delta) := \{y \in \mathbb{R}^d : |y - x| < \delta\}$ we denote the ball centered at $x \in \mathbb{R}^d$ with radius $\delta > 0$. Given subsets A and B of \mathbb{R}^d , and a function ϕ on set B , we set $\phi_A := \phi \chi_{A \cap B}$. We let $\phi_{x, \delta} := \phi_{B(x, \delta)}$.

Theorem 1.1. *Let Ω be an open subset of \mathbb{R}^d , $K = K_1 + K_2$ with $K_1, K_2 \geq 0$, and consider a real measurable function $V = V^{(1)} + V^{(2)}$ on Ω with $\|V^{(1)}\|_\infty \leq K_1$. Let $\psi \in L^2(\Omega)$ be real valued with $\Delta\psi \in L^2_{loc}(\Omega)$, and suppose*

$$(1.5) \quad \zeta = -\Delta\psi + V\psi \in L^2(\Omega).$$

Fix a bounded measurable set $\Theta \subset \Omega$ where $\|\psi_\Theta\|_2 > 0$, and set

$$(1.6) \quad Q(x, \Theta) := \sup_{y \in \Theta} |y - x| \quad \text{for } x \in \Omega.$$

Consider $x_0 \in \Omega \setminus \overline{\Theta}$ such that

$$(1.7) \quad Q = Q(x_0, \Theta) \geq 1 \quad \text{and} \quad B(x_0, 6Q + 2) \subset \Omega,$$

and take

$$(1.8) \quad 0 < \delta \leq \min\{\text{dist}(x_0, \Theta), \frac{1}{2}\}.$$

There is a constant $m_d > 0$, depending only on d , such that:

- (i) *If either $d \geq 3$ and $\|V^{(2)}\|_p \leq K_2$ with $p \geq d$, or $d = 2$ and $(\| |V^{(2)}|^p \|_{\varphi^*})^{\frac{1}{p}} \leq K_2$ with $p \geq 2$, we have*

$$(1.9) \quad \left(\frac{\delta}{Q}\right)^{m_d(1+K^{\frac{2p}{3p-2d}})(Q^{\frac{4p-2d}{3p-2d}} + \log \frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2})} \|\psi_\Theta\|_2^2 \leq \|\psi_{x_0, \delta}\|_2^2 + \delta^2 \|\zeta_\Omega\|_2^2.$$

In particular, if $d = 2$ it suffices to require $\|V^{(2)}\|_p \leq K_2$ with $p > 2$ to obtain (1.9).

- (ii) *If $d = 1$ and $\|V^{(2)}\|_p \leq K_2$ with $p \geq 2$, we have*

$$(1.10) \quad \left(\frac{\delta}{Q}\right)^{m_1(1+K^{\frac{2p}{3p-4}})(Q^{\frac{4p-4}{3p-4}} + \log \frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2})} \|\psi_\Theta\|_2^2 \leq \|\psi_{x_0, \delta}\|_2^2 + \delta^2 \|\zeta_\Omega\|_2^2.$$

Letting $p \rightarrow \infty$ in Theorem 1.1 we recover [BK1, Theorem 3.4]. The proof of Theorem 1.1, given in Section 2, relies on a Carleman estimate of Escoriaza and Vesella [EV, Theorem 2], stated in Lemma 2.1. To control singular potentials we use all the terms in this estimate, including the the gradient term, and Sobolev's inequalities. In the proofs for bounded potentials [BK, GK2, BK1] it suffices to use a simpler version of this Carleman estimate without the the gradient term (see [BK, Lemma 3.15]).

As an application of Theorem 1.1, we prove a unique continuation principle for spectral projections of Schrödinger operators with singular potentials, extending [K1, Theorem 1.1] (in the form given in [KLN, Theorem B.1]) to Schrödinger operators with singular potentials. (See also [CHK1, Section 4], [CHK2, Theorem 2.1],

[GK2, Theorem A.6], and [RoV, Theorem 2.1] for unique continuation principles for spectral projections of Schrödinger operators with bounded potentials.)

We consider rectangles in \mathbb{R}^d of the form

$$(1.11) \quad \Lambda = \Lambda_{\mathbf{L}}(a) = a + \prod_{j=1}^d \left(-\frac{L_j}{2}, \frac{L_j}{2}\right) = \prod_{j=1}^d \left(a_j - \frac{L_j}{2}, a_j + \frac{L_j}{2}\right),$$

where $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ and $\mathbf{L} = (L_1, \dots, L_d) \in (0, \infty)^d$. (We write $\Lambda_L(a) = \Lambda_{\mathbf{L}}(a)$ in the special case $L_j = L$ for $j = 1, \dots, d$.) Given a Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$, by $H_\Lambda = -\Delta_\Lambda + V_\Lambda$ we denote the restriction of H to the rectangle Λ with either Dirichlet or periodic boundary condition: Δ_Λ is the Laplacian on Λ with either Dirichlet or periodic boundary condition, and V_Λ is the restriction of V to Λ .

Theorem 1.2. *Let $H = -\Delta + V$ be a Schrödinger operator on $L^2(\mathbb{R}^d)$, where $V = V^{(1)} + V^{(2)}$ with $\|V^{(1)}\|_\infty \leq K_1 < \infty$ and $\|V^{(2)}\|_p \leq K_2 < \infty$ with $p \geq d$ for $d \geq 3$, $p > 2$ for $d = 2$, and $p \geq 2$ for $d = 1$. Set $K = K_1 + K_2$. Fix $\delta \in (0, \frac{1}{2}]$, and let $\{y_k\}_{k \in \mathbb{Z}^d}$ be sites in \mathbb{R}^d with $B(y_k, \delta) \subset \Lambda_1(k)$ for all $k \in \mathbb{Z}^d$. There exists a constant $M_d > 0$, depending only on d , such that, defining $\gamma = \gamma(d, p, K, \delta, E_0) > 0$ for $E_0 > 0$ by*

$$(1.12) \quad \gamma^2 = \begin{cases} \frac{1}{2} \delta^{M_d \left(1 + (K+E_0) \frac{4p^2}{(3p-2d)(2p-d)}\right)} & \text{for } d \geq 2, \\ \frac{1}{2} \delta^{M_d \left(1 + (K+E_0) \frac{2p^2}{(3p-4)(p-1)}\right)} & \text{for } d = 1 \end{cases},$$

then, given a rectangle Λ as in (1.11), where $a \in \mathbb{R}^d$ and $L_j \geq 114\sqrt{d}$ for $j = 1, \dots, d$, and a closed interval $I \subset (-\infty, E_0]$ with $|I| \leq 2\gamma$, we have

$$(1.13) \quad \chi_I(H_\Lambda) W^{(\Lambda)} \chi_I(H_\Lambda) \geq \gamma^2 \chi_I(H_\Lambda),$$

where

$$(1.14) \quad W^{(\Lambda)} = \sum_{k \in \mathbb{Z}^d, \Lambda_1(k) \subset \Lambda} \chi_{B(y_k, \delta)}.$$

The proof of Theorem 1.2 is discussed in Section 3.

Remark 1.3. Using Theorem 1.2 we can prove optimal Wegner estimates for Anderson Hamiltonians with singular background potentials, extending the results of [KL].

2. QUANTITATIVE UNIQUE CONTINUATION PRINCIPLE FOR SCHRÖDINGER OPERATORS WITH SINGULAR POTENTIALS

The proof of Theorem 1.1 is based on a Carleman estimate of Escauriaza and Vesella [EV, Theorem 2], which we state in a ball of radius $\varrho > 0$.

Lemma 2.1. *Given $\varrho > 0$, the function $\omega_\varrho(x) = \phi(\frac{1}{\varrho}|x|)$ on \mathbb{R}^d , where $\phi(s) := se - \int_0^s \frac{1-e^{-t}}{t} dt$, is a strictly increasing continuous function on $[0, \infty)$, C^∞ on $(0, \infty)$, satisfying*

$$(2.1) \quad \frac{1}{C_1 \varrho} |x| \leq \omega_\varrho(x) \leq \frac{1}{\varrho} |x| \quad \text{for } x \in B(0, \varrho),$$

where $C_1 = \phi(1)^{-1} \in (2, 3)$. Moreover, there exist positive constants C_2 and C_3 , depending only on d , such that for all $\alpha \geq C_2$ and all real valued functions $f \in H^2(B(0, \varrho))$ with $\text{supp } f \subset B(0, \varrho) \setminus \{0\}$ we have

$$(2.2) \quad \alpha^3 \int_{\mathbb{R}^d} \omega_\varrho^{-1-2\alpha} f^2 dx + \alpha \varrho^2 \int_{\mathbb{R}^d} \omega_\varrho^{1-2\alpha} |\nabla f|^2 dx \leq C_3 \varrho^4 \int_{\mathbb{R}^d} \omega_\varrho^{2-2\alpha} (\Delta f)^2 dx.$$

This estimate is given in the parabolic setting in [EV], but the estimate in the elliptic setting as in the lemma follows immediately by the argument in [KSU, Proposition B.3]. In the proofs of the quantitative unique continuation principle for bounded potentials [BK, GK2, BK1] only the first term in the left hand side of (2.2) is used (see [BK, Lemma 3.15]), but for singular potentials we also need to use the gradient term in the left hand side of (2.2) and Sobolev's inequalities.

Proof of Theorem 1.1. Let C_1, C_2, C_3 be the constants of Lemma 2.1, which depend only on d . Without loss of generality $C_2 > 1$. By C_j , $j = 4, 5, \dots$, we will always denote an appropriate nonzero constant depending only on d .

We follow Bourgain and Klein's proof for bounded potentials [BK1, Theorem 3.4]. Let $x_0 \in \Omega \setminus \overline{\Theta}$ be as in (1.7). Without loss of generality we take $x_0 = 0$, $\Theta \subset B(0, 2C_1Q)$, and $\Omega = B(0, \varrho)$, where $\varrho = 2C_1Q + 2$, and let δ be as in (1.8). Proceeding as in [BK1, Theorem 3.4], we fix a function $\eta \in C_c^\infty(\mathbb{R}^d)$ given by $\eta(x) = \xi(|x|)$, where ξ is an even C^∞ function on \mathbb{R} , $0 \leq \xi \leq 1$, such that

$$(2.3) \quad \begin{aligned} \xi(s) &= 1 \quad \text{if } \frac{3}{4}\delta \leq |s| \leq 2C_1Q, \quad \xi(s) = 0 \quad \text{if } |s| \leq \frac{1}{4}\delta \text{ or } |s| \geq 2C_1Q + 1, \\ |\xi^j(s)| &\leq \left(\frac{4}{\delta}\right)^j \quad \text{if } |s| \leq \frac{3}{4}\delta, \quad |\xi^j(s)| \leq 2^j \quad \text{if } |s| \geq 2C_1Q, j = 1, 2, \\ |\nabla \eta(x)| &\leq \sqrt{d} |\xi'(|x|)| \quad \text{and} \quad |\Delta \eta(x)| \leq d |\xi''(|x|)|, \\ \text{supp } \nabla \eta &\subset \{\frac{\delta}{4} \leq |x| \leq \frac{3\delta}{4}\} \cup \{2C_1Q \leq |x| \leq 2C_1Q + 1\}. \end{aligned}$$

Let $\alpha \geq C_2$. Applying Lemma 2.1 to the function $\eta\psi$ gives

$$(2.4) \quad \begin{aligned} &\frac{\alpha^3}{3C_3\varrho^4} \int_{\mathbb{R}^d} \omega_\varrho^{-1-2\alpha} \eta^2 \psi^2 dx + \frac{\alpha}{3C_3\varrho^2} \int_{\mathbb{R}^d} \omega_\varrho^{1-2\alpha} |\nabla(\eta\psi)|^2 dx \\ &\leq \frac{1}{3} \int_{\mathbb{R}^d} \omega_\varrho^{2-2\alpha} (\Delta(\eta\psi))^2 dx \leq \int_{\mathbb{R}^d} \omega_\varrho^{2-2\alpha} \eta^2 (\Delta\psi)^2 dx \\ &\quad + 4 \int_{\text{supp } \nabla \eta} \omega_\varrho^{2-2\alpha} |\nabla \eta|^2 |\nabla \psi|^2 dx + \int_{\text{supp } \nabla \eta} \omega_\varrho^{2-2\alpha} (\Delta \eta)^2 \psi^2 dx. \end{aligned}$$

Using (1.5), $\|V^{(1)}\|_\infty \leq K_1$, and $\omega_\varrho \leq 1$ on $\text{supp } \eta$, we have

$$(2.5) \quad \begin{aligned} \int_{\mathbb{R}^d} \omega_\varrho^{2-2\alpha} \eta^2 (\Delta\psi)^2 dx &\leq 2 \int_{\mathbb{R}^d} V^2 \omega_\varrho^{2-2\alpha} \eta^2 \psi^2 dx + 2 \int_{\mathbb{R}^d} \omega_\varrho^{2-2\alpha} \eta^2 \zeta^2 dx \\ &\leq 4K_1^2 \int_{\mathbb{R}^d} \omega_\varrho^{-1-2\alpha} \eta^2 \psi^2 dx + 4 \int_{\mathbb{R}^d} (V^{(2)})^2 \omega_\varrho^{2-2\alpha} \eta^2 \psi^2 dx + 2 \int_{\mathbb{R}^d} \omega_\varrho^{2-2\alpha} \eta^2 \zeta^2 dx. \end{aligned}$$

Given $M > 0$, we write $V^{(2)} = U_M + W_M$, where $U_M = V^{(2)} \chi_{\{|V^{(2)}| \leq \sqrt{M}\}}$ and $W_M = V^{(2)} \chi_{\{|V^{(2)}| > \sqrt{M}\}}$. We have

$$(2.6) \quad \int_{\mathbb{R}^d} (V^{(2)})^2 \omega_\varrho^{2-2\alpha} \eta^2 \psi^2 dx \leq M \int_{\mathbb{R}^d} \omega_\varrho^{-1-2\alpha} \eta^2 \psi^2 dx + \int_{\mathbb{R}^d} W_M^2 \omega_\varrho^{2-2\alpha} \eta^2 \psi^2 dx.$$

Combining (2.4), (2.5) and (2.6), we have

$$\begin{aligned}
 (2.7) \quad & \left(\frac{\alpha^3}{3C_3\varrho^4} - 4K_1^2 - 4M \right) \int_{\mathbb{R}^d} \omega_\varrho^{-1-2\alpha} \eta^2 \psi^2 dx + \frac{\alpha}{3C_3\varrho^2} \int_{\mathbb{R}^d} \omega_\varrho^{1-2\alpha} |\nabla(\eta\psi)|^2 dx \\
 & \leq 4 \int_{\mathbb{R}^d} W_M^2 \omega_\varrho^{2-2\alpha} \eta^2 \psi^2 dx + 2 \int_{\mathbb{R}^d} \omega_\varrho^{2-2\alpha} \eta^2 \zeta^2 dx \\
 & \quad + 4 \int_{\text{supp } \nabla \eta} \omega_\varrho^{2-2\alpha} |\nabla \eta|^2 |\nabla \psi|^2 dx + \int_{\text{supp } \nabla \eta} \omega_\varrho^{2-2\alpha} (\Delta \eta)^2 \psi^2 dx.
 \end{aligned}$$

Note that for $1 \leq q \leq p$ we have

$$(2.8) \quad \|W_M\|_q \leq M^{-\frac{p-q}{2q}} \|W_M\|_p^{\frac{p}{q}} \leq M^{-\frac{p-q}{2q}} \|V^{(2)}\|_p^{\frac{p}{q}} \leq M^{-\frac{p-q}{2q}} K_2^{\frac{p}{q}}.$$

We set $K = K_1 + K_2$ with $K_1, K_2 \geq 0$.

We consider three cases:

(a) $d \geq 3$: Let $\|V^{(2)}\|_p \leq K_2$ with $p \geq d$. Using Hölder's inequality and (2.8) with $q = d$, we get

$$\begin{aligned}
 (2.9) \quad & \int_{\mathbb{R}^d} W_M^2 \omega_\varrho^{2-2\alpha} \eta^2 \psi^2 dx \leq \|W_M^2\|_{\frac{d}{2}} \|\omega_\varrho^{2-2\alpha} \eta^2 \psi^2\|_{\frac{d}{d-2}} \\
 & = \|W_M\|_d^2 \|\omega_\varrho^{1-\alpha} \eta \psi\|_{\frac{2d}{d-2}}^2 \leq M^{-\frac{p-d}{d}} K_2^{\frac{2p}{d}} \|\omega_\varrho^{1-\alpha} \eta \psi\|_{\frac{2d}{d-2}}^2.
 \end{aligned}$$

Using Sobolev's inequality (e.g., [GiT, Theorem 7.10]), we get

$$\begin{aligned}
 (2.10) \quad & \|\omega_\varrho^{1-\alpha} \eta \psi\|_{\frac{2d}{d-2}}^2 \leq C_4 \left(\int_{\mathbb{R}^d} |\nabla(\omega_\varrho^{1-\alpha} \eta \psi)|^2 \right) \\
 & \leq 2C_4 \int_{\mathbb{R}^d} |\nabla \omega_\varrho^{1-\alpha}|^2 \eta^2 \psi^2 dx + 2C_4 \int_{\mathbb{R}^d} \omega_\varrho^{1-2\alpha} |\nabla(\eta\psi)|^2 dx.
 \end{aligned}$$

Since

$$(2.11) \quad |\nabla \omega_\varrho^{1-\alpha}|^2 = (1-\alpha)^2 \frac{\omega_\varrho^{2-2\alpha}}{|x|^2 \exp(\frac{2}{\varrho}|x|)} \leq \frac{\alpha^2}{\varrho^2} \omega_\varrho^{-2\alpha},$$

we have (recall $\omega_\varrho \leq 1$ on $\text{supp } \eta$)

$$(2.12) \quad \int_{\mathbb{R}^d} |\nabla \omega_\varrho^{1-\alpha}|^2 \eta^2 \psi^2 dx \leq \frac{\alpha^2}{\varrho^2} \int_{\mathbb{R}^d} \omega_\varrho^{-1-2\alpha} \eta^2 \psi^2 dx.$$

Combining (2.7), (2.9), (2.10) and (2.12), we conclude that

$$\begin{aligned}
 (2.13) \quad & \left(\frac{\alpha^3}{3C_3\varrho^4} - 4K_1^2 - 4M - 8C_4 M^{-\frac{p-d}{d}} K_2^{\frac{2p}{d}} \frac{\alpha^2}{\varrho^2} \right) \int_{\mathbb{R}^d} \omega_\varrho^{-1-2\alpha} \eta^2 \psi^2 dx \\
 & \quad + \left(\frac{\alpha}{3C_3\varrho^2} - 8C_4 M^{-\frac{p-d}{d}} K_2^{\frac{2p}{d}} \right) \int_{\mathbb{R}^d} \omega_\varrho^{1-2\alpha} |\nabla(\eta\psi)|^2 dx \\
 & \leq 4 \int_{\text{supp } \nabla \eta} \omega_\varrho^{2-2\alpha} |\nabla \eta|^2 |\nabla \psi|^2 dx + \int_{\text{supp } \nabla \eta} \omega_\varrho^{2-2\alpha} (\Delta \eta)^2 \psi^2 dx \\
 & \quad + 2 \int_{\text{supp } \eta} \omega_\varrho^{2-2\alpha} \eta^2 \zeta^2 dx.
 \end{aligned}$$

Assuming $\alpha \geq \varrho$ and setting $M = K_2^2 \alpha^{\frac{2d}{p}} \varrho^{\frac{-2d}{p}}$, we have

$$(2.14) \quad \begin{aligned} 4K_1^2 + 4M + 8C_4 M^{-\frac{p-d}{d}} K_2^{\frac{2p}{d}} \alpha^2 \varrho^{-2} &= 4K_1^2 + 4K_2^2(1 + 2C_4) \alpha^{\frac{2d}{p}} \varrho^{\frac{-2d}{p}} \\ &\leq (4K^2(1 + 2C_4)) \alpha^{\frac{2d}{p}} \varrho^{\frac{-2d}{p}}. \end{aligned}$$

Taking

$$(2.15) \quad \alpha \geq C_5(1 + K^{\frac{2p}{3p-2d}}) \varrho^{\frac{4p-2d}{3p-2d}} \geq C_5(1 + K^{\frac{2p}{3p-2d}}) \varrho^{\frac{4}{3}},$$

we can guarantee that $\alpha > C_2$,

$$(2.16) \quad \frac{\alpha^3}{3C_3 \varrho^4} \geq 3(4K^2(1 + 2C_4) \alpha^{\frac{2d}{p}} \varrho^{\frac{-2d}{p}}),$$

and

$$(2.17) \quad \frac{\alpha}{3C_3 \varrho^2} - 8C_4 M^{-\frac{p-d}{d}} K_2^{\frac{2p}{d}} \geq 0.$$

Using (2.1) and recalling (1.6), we obtain

$$(2.18) \quad \int_{\mathbb{R}^d} \omega_{\varrho}^{-1-2\alpha} \eta^2 \psi^2 dx \geq \left(\frac{\varrho}{Q} \right)^{1+2\alpha} \|\psi_{\Theta}\|_2^2 \geq (2C_1)^{1+2\alpha} \|\psi_{\Theta}\|_2^2.$$

Combining (2.13), (2.16), (2.17) and (2.18), we conclude that

$$(2.19) \quad \begin{aligned} \frac{2\alpha^3}{9C_3 \varrho^4} (2C_1)^{1+2\alpha} \|\psi_{\Theta}\|_2^2 &\leq 4 \int_{\text{supp } \nabla \eta} \omega_{\varrho}^{2-2\alpha} |\nabla \eta|^2 |\nabla \psi|^2 dx \\ &\quad + \int_{\text{supp } \nabla \eta} \omega_{\varrho}^{2-2\alpha} (\Delta \eta)^2 \psi^2 dx + 2 \int_{\text{supp } \eta} \omega_{\varrho}^{2-2\alpha} \eta^2 \zeta^2 dx. \end{aligned}$$

Let $f \in \mathcal{D}(\nabla)$. For arbitrary $M > 0$ we have

$$(2.20) \quad \left| \int_{\mathbb{R}^d} V f^2 dx \right| \leq (K_1 + M^{\frac{1}{2}}) \|f\|_2^2 + \int_{\mathbb{R}^d} |W_M| f^2 dx.$$

Using Hölder's inequality, (2.8) with $q = \frac{d}{2}$, and Sobolev's inequality, we get

$$(2.21) \quad \left| \int_{\mathbb{R}^d} V f^2 dx \right| \leq (K_1 + M^{\frac{1}{2}}) \|f\|_2^2 + C_4 M^{-\frac{2p-d}{2d}} K_2^{\frac{2p}{d}} \|\nabla f\|_2^2.$$

Taking $M = (2C_4 K_2^{\frac{2p}{d}})^{\frac{2d}{2p-d}}$ (we can require $C_4 \geq 1$), we get

$$(2.22) \quad \left| \int_{\mathbb{R}^d} V f^2 dx \right| \leq 2C_4(1 + K^{\frac{2p}{2p-d}}) \|f\|_2^2 + \frac{1}{2} \|\nabla f\|_2^2.$$

We have

$$(2.23) \quad \begin{aligned} &\int_{\{2C_1 Q \leq |x| \leq 2C_1 Q+1\}} \omega_{\varrho}^{2-2\alpha} (4|\nabla \eta|^2 |\nabla \psi|^2 + (\Delta \eta)^2 \psi^2) dx \\ &\leq 16d^2 \left(\frac{C_1 \varrho}{2C_1 Q} \right)^{2\alpha-2} \int_{\{2C_1 Q \leq |x| \leq 2C_1 Q+1\}} (4|\nabla \psi|^2 + \psi^2) dx \\ &\leq C_6 \left(\frac{5}{4} C_1 \right)^{2\alpha-2} \int_{\{2C_1 Q-1 \leq |x| \leq 2C_1 Q+2\}} (\zeta^2 + (1 + K^{\frac{2p}{2p-d}}) \psi^2) dx \\ &\leq C_6 \left(\frac{5}{4} C_1 \right)^{2\alpha-2} (\|\zeta_{\Omega}\|_2^2 + (1 + K^{\frac{2p}{2p-d}}) \|\psi_{\Omega}\|_2^2), \end{aligned}$$

where we used (2.22) and an interior estimate (e.g., [GK1, Lemma A.2]). Similarly,

$$\begin{aligned}
 (2.24) \quad & \int_{\{\frac{\delta}{4} \leq |x| \leq \frac{3\delta}{4}\}} \omega_\varrho^{2-2\alpha} (4|\nabla \eta|^2 |\nabla \psi|^2 + (\Delta \eta)^2 \psi^2) dx \\
 & \leq 256d^2 \delta^{-4} (4\delta^{-1} C_1 \varrho)^{2\alpha-2} \int_{\{\frac{\delta}{4} \leq |x| \leq \frac{3\delta}{4}\}} (4|\nabla \psi|^2 + \psi^2) dx \\
 & \leq C_7 \delta^{-4} (4\delta^{-1} C_1 \varrho)^{2\alpha-2} \int_{\{|x| \leq \delta\}} (\zeta^2 + (K^{\frac{2p}{2p-d}} + \delta^{-2}) \psi^2) dx \\
 & \leq C_7 \delta^{-4} (16\delta^{-1} C_1^2 Q)^{2\alpha-2} (\|\zeta_\Omega\|_2^2 + (K^{\frac{2p}{2p-d}} + \delta^{-2}) \|\psi_{0,\delta}\|_2^2).
 \end{aligned}$$

In addition,

$$(2.25) \quad \int_{\text{supp } \eta} \omega_\varrho^{2-2\alpha} \eta^2 \zeta^2 dx \leq (4\delta^{-1} C_1 \varrho)^{2\alpha-2} \|\zeta_\Omega\|_2^2 \leq (16\delta^{-1} C_1^2 Q)^{2\alpha-2} \|\zeta_\Omega\|_2^2.$$

If we have

$$(2.26) \quad \frac{\alpha^3}{\varrho^4} \left(\frac{8}{5}\right)^{2\alpha} \|\psi_\Theta\|_2^2 \geq C_8 (1 + K^{\frac{2p}{2p-d}}) \|\psi_\Omega\|_2^2,$$

we obtain

$$(2.27) \quad C_6 \left(\frac{5}{4} C_1\right)^{2\alpha-2} (1 + K^{\frac{2p}{2p-d}}) \|\psi_\Omega\|_2^2 \leq \frac{1}{2} \frac{2\alpha^3}{9C_3\varrho^4} (2C_1)^{1+2\alpha} \|\psi_\Theta\|_2^2,$$

so we conclude that

$$\begin{aligned}
 (2.28) \quad & \frac{\alpha^3}{9C_3\varrho^4} (2C_1)^{1+2\alpha} \|\psi_\Theta\|_2^2 \\
 & \leq C_9 \delta^{-4} (16\delta^{-1} C_1^2 Q)^{2\alpha-2} ((K^{\frac{2p}{2p-d}} + \delta^{-2}) \|\psi_{0,\delta}\|_2^2 + \|\zeta_\Omega\|_2^2).
 \end{aligned}$$

Thus,

$$(2.29) \quad \frac{\alpha^3}{\varrho^4} Q^4 ((8C_1 Q)^{-1} \delta)^{2\alpha+2} \|\psi_\Theta\|_2^2 \leq C_{10} ((K^{\frac{2p}{2p-d}} + \delta^{-2}) \|\psi_{0,\delta}\|_2^2 + \|\zeta_\Omega\|_2^2).$$

Since $(\frac{\delta}{Q})^5 \leq (\frac{1}{2})^5 \leq \frac{1}{8C_1}$ by (1.8), we have

$$(2.30) \quad \frac{\alpha^3}{\varrho^4} Q^6 \left(\frac{\delta}{Q}\right)^{12\alpha+14} \|\psi_\Theta\|_2^2 \leq C_{11} ((1 + K^{\frac{2p}{2p-d}}) \|\psi_{0,\delta}\|_2^2 + \delta^2 \|\zeta_\Omega\|_2^2).$$

To satisfy (2.15) and (2.26), we choose

$$(2.31) \quad \alpha = C_{12} (1 + K^{\frac{2p}{3p-2d}}) \left(Q^{\frac{4p-2d}{3p-2d}} + \log \frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2} \right),$$

Combining with (2.30), and recalling $Q \geq 1$, we get

$$\begin{aligned}
 (2.32) \quad & (1 + K^{\frac{2p}{3p-2d}})^3 \left(\frac{\delta}{Q}\right)^{C_{13}(1 + K^{\frac{2p}{3p-2d}}) \left(Q^{\frac{4p-2d}{3p-2d}} + \log \frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2} \right)} \|\psi_\Theta\|_2^2 \\
 & \leq C_{14} ((1 + K^{\frac{2p}{2p-d}}) \|\psi_{0,\delta}\|_2^2 + \delta^2 \|\zeta_\Omega\|_2^2),
 \end{aligned}$$

and hence

$$(2.33) \quad \left(\frac{\delta}{Q}\right)^{m_d(1 + K^{\frac{2p}{3p-2d}}) \left(Q^{\frac{4p-2d}{3p-2d}} + \log \frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2} \right)} \|\psi_\Theta\|_2^2 \leq \|\psi_{x_0,\delta}\|_2^2 + \delta^2 \|\zeta_\Omega\|_2^2,$$

where $m_d > 0$ is a constant depending only on d .

(b) $d = 2$: Let $(\|V^{(2)}\|^p)_{\varphi^*}^{\frac{1}{p}} \leq K_2$ with $p \geq 2$. Given $K_2 > 0$ and $M > 0$, we have

$$(2.34) \quad \int_{\mathbb{R}^2} \varphi^* \left(\frac{|W_M^2|}{M^{-\frac{p-2}{2}} K_2^p} \right) dx \leq \int_{\mathbb{R}^2} \varphi^* \left(\frac{|V^{(2)}|^p}{K_2^p} \right) dx,$$

and hence, using $\|V^{(2)}\|_{\varphi^*} \leq K_2^p$, we get

$$(2.35) \quad \|W_M^2\|_{\varphi^*} \leq M^{-\frac{p-2}{2}} K_2^p.$$

Using Hölder's inequality for Orlicz spaces (1.3), and (2.35), we get

$$(2.36) \quad \begin{aligned} \int_{\mathbb{R}^2} W_M^2 \omega_\varrho^{2-2\alpha} \eta^2 \psi^2 dx &\leq 2 \|W_M^2\|_{\varphi^*} \|\omega_\varrho^{2-2\alpha} \eta^2 \psi^2\|_\varphi \\ &\leq 2 M^{-\frac{p-2}{2}} K_2^p \|\omega_\varrho^{2-2\alpha} \eta^2 \psi^2\|_\varphi. \end{aligned}$$

Using the Sobolev inequality given in [AT, Theorem 0.1], we obtain

$$(2.37) \quad \begin{aligned} \|\omega_\varrho^{2-2\alpha} \eta^2 \psi^2\|_\varphi &\leq C_4 \left(\int_{\mathbb{R}^2} |\omega_\varrho^{1-\alpha} \eta \psi|^2 dx + \int_{\mathbb{R}^2} |\nabla(\omega_\varrho^{1-\alpha} \eta \psi)|^2 dx \right) \\ &\leq C_4 \int_{\mathbb{R}^2} |\omega_\varrho^{1-\alpha} \eta \psi|^2 dx + 2C_4 \int_{\mathbb{R}^2} |\nabla \omega_\varrho^{1-\alpha}|^2 \eta^2 \psi^2 dx \\ &\quad + 2C_4 \int_{\mathbb{R}^2} \omega_\varrho^{1-2\alpha} |\nabla(\eta \psi)|^2 dx. \end{aligned}$$

Combining (2.7), (2.36), (2.37), and (2.12) with $d = 2$, we conclude that

$$(2.38) \quad \begin{aligned} &\left(\frac{\alpha^3}{3C_3\varrho^4} - 4K_1^2 - 4M - 8C_4M^{-\frac{p-2}{2}}K_2^p - 16C_4M^{-\frac{p-2}{2}}K_2^p\frac{\alpha^2}{\varrho^2} \right) \int_{\mathbb{R}^2} \omega_\varrho^{-1-2\alpha} \eta^2 \psi^2 dx \\ &\quad + \left(\frac{\alpha}{3C_3\varrho^2} - 16C_4M^{-\frac{p-2}{2}}K_2^p \right) \int_{\mathbb{R}^2} \omega_\varrho^{1-2\alpha} |\nabla(\eta \psi)|^2 dx \\ &\leq 4 \int_{\text{supp } \nabla \eta} \omega_\varrho^{2-2\alpha} |\nabla \eta|^2 |\nabla \psi|^2 dx + \int_{\text{supp } \nabla \eta} \omega_\varrho^{2-2\alpha} (\Delta \eta)^2 \psi^2 dx \\ &\quad + 2 \int_{\text{supp } \eta} \omega_\varrho^{2-2\alpha} \eta^2 \zeta^2 dx. \end{aligned}$$

Assuming $\alpha \geq \varrho$ and setting $M = K_2^2 \alpha^{\frac{4}{p}} \varrho^{-\frac{4}{p}}$, we have

$$(2.39) \quad \begin{aligned} &4K_1^2 + 4M + 8C_4M^{-\frac{p-2}{2}}K_2^p + 16C_4M^{-\frac{p-2}{2}}K_2^p\frac{\alpha^2}{\varrho^2} \\ &\leq 4K_1^2 + 4M + 24C_4M^{-\frac{p-2}{2}}K_2^p\frac{\alpha^2}{\varrho^2} \\ &= 4K_1^2 + 4K_2^2(1 + 6C_4)\alpha^{\frac{4}{p}}\varrho^{-\frac{4}{p}} \leq 4K^2(1 + 6C_4)\alpha^{\frac{4}{p}}\varrho^{-\frac{4}{p}}. \end{aligned}$$

Taking

$$(2.40) \quad \alpha \geq C_5(1 + K^{\frac{2p}{3p-4}})\varrho^{\frac{4p-4}{3p-4}} \geq C_5(1 + K^{\frac{2p}{3p-4}})\varrho^{\frac{4}{3}},$$

we can guarantee that $\alpha > C_2$,

$$(2.41) \quad \frac{\alpha^3}{3C_3\varrho^4} \geq 3(4K^2(1 + 6C_4)\alpha^{\frac{4}{p}}\varrho^{-\frac{4}{p}}),$$

and

$$(2.42) \quad \frac{\alpha}{3C_3\varrho^2} - 16C_4M^{-\frac{p-2}{2}}K_2^p \geq 0.$$

Using (2.1) and recalling (1.6), we obtain

$$(2.43) \quad \int_{\mathbb{R}^2} \omega_\varrho^{-1-2\alpha} \eta^2 \psi^2 dx \geq \left(\frac{\varrho}{Q}\right)^{1+2\alpha} \|\psi_\Theta\|_2^2 \geq (2C_1)^{1+2\alpha} \|\psi_\Theta\|_2^2.$$

Combining (2.38), (2.41), (2.42) and (2.43), we conclude that

$$(2.44) \quad \begin{aligned} \frac{2\alpha^3}{9C_3\varrho^4} (2C_1)^{1+2\alpha} \|\psi_\Theta\|_2^2 &\leq 4 \int_{\text{supp } \nabla \eta} \omega_\varrho^{2-2\alpha} |\nabla \eta|^2 |\nabla \psi|^2 dx \\ &+ \int_{\text{supp } \nabla \eta} \omega_\varrho^{2-2\alpha} (\Delta \eta)^2 \psi^2 dx + 2 \int_{\text{supp } \eta} \omega_\varrho^{2-2\alpha} \eta^2 \zeta^2 dx. \end{aligned}$$

Given $M > 0$, we have

$$(2.45) \quad \int_{\mathbb{R}^2} \varphi^* \left(\frac{|W_M|}{M^{-\frac{p-1}{2}} K_2^p} \right) dx \leq \int_{\mathbb{R}^2} \varphi^* \left(\frac{|V^{(2)}|^p}{K_2^p} \right) dx,$$

and hence, using $\| |V^{(2)}|^p \|_{\varphi^*} \leq K_2^p$, we get $\|W_M\|_{\varphi^*} \leq M^{-\frac{p-1}{2}} K_2^p$. Let $f \in \mathcal{D}(\nabla)$. Then, using (2.20), Hölder's inequality for Orlicz spaces (1.3), and the Sobolev inequality in [AT, Theorem 0.1], we get

$$(2.46) \quad \left| \int_{\mathbb{R}^2} V f^2 dx \right| \leq (K_1 + M^{\frac{1}{2}} + 2C_4 M^{-\frac{p-1}{2}} K_2^p) \|f\|_2^2 + 2C_4 M^{-\frac{p-1}{2}} K_2^p \|\nabla f\|_2^2.$$

Taking $M = (4C_4 K_2^p)^{\frac{2}{p-1}}$ (we can require $C_4 \geq 1$), we get

$$(2.47) \quad \left| \int_{\mathbb{R}^2} V f^2 dx \right| \leq 4C_4 (1 + K^{\frac{p}{p-1}}) \|f\|_2^2 + \frac{1}{2} \|\nabla f\|_2^2.$$

We have

$$(2.48) \quad \begin{aligned} &\int_{\{2C_1 Q \leq |x| \leq 2C_1 Q+1\}} \omega_\varrho^{2-2\alpha} (4|\nabla \eta|^2 |\nabla \psi|^2 + (\Delta \eta)^2 \psi^2) dx \\ &\leq 64 \left(\frac{C_1 \varrho}{2C_1 Q} \right)^{2\alpha-2} \int_{\{2C_1 Q \leq |x| \leq 2C_1 Q+1\}} (4|\nabla \psi|^2 + \psi^2) dx \\ &\leq C_6 \left(\frac{5}{4} C_1 \right)^{2\alpha-2} \int_{\{2C_1 Q-1 \leq |x| \leq 2C_1 Q+2\}} (\zeta^2 + (1 + K^{\frac{p}{p-1}}) \psi^2) dx \\ &\leq C_6 \left(\frac{5}{4} C_1 \right)^{2\alpha-2} (\|\zeta_\Omega\|_2^2 + (1 + K^{\frac{p}{p-1}}) \|\psi_\Omega\|_2^2), \end{aligned}$$

where we used (2.47) and an interior estimate. Similarly,

$$(2.49) \quad \begin{aligned} &\int_{\{\frac{\delta}{4} \leq |x| \leq \frac{3\delta}{4}\}} \omega_\varrho^{2-2\alpha} (4|\nabla \eta|^2 |\nabla \psi|^2 + (\Delta \eta)^2 \psi^2) dx \\ &\leq 1024\delta^{-4} (4\delta^{-1} C_1 \varrho)^{2\alpha-2} \int_{\{\frac{\delta}{4} \leq |x| \leq \frac{3\delta}{4}\}} (4|\nabla \psi|^2 + \psi^2) dx \\ &\leq C_7 \delta^{-4} (4\delta^{-1} C_1 \varrho)^{2\alpha-2} \int_{\{|x| \leq \delta\}} (\zeta^2 + (K^{\frac{p}{p-1}} + \delta^{-2}) \psi^2) dx \\ &\leq C_7 \delta^{-4} (16\delta^{-1} C_1^2 Q)^{2\alpha-2} (\|\zeta_\Omega\|_2^2 + (K^{\frac{p}{p-1}} + \delta^{-2}) \|\psi_{0,\delta}\|_2^2). \end{aligned}$$

In addition,

$$(2.50) \quad \int_{\text{supp } \eta} \omega_{\varrho}^{2-2\alpha} \eta^2 \zeta^2 dx \leq (4\delta^{-1} C_1 \varrho)^{2\alpha-2} \|\zeta_{\Omega}\|_2^2 \leq (16\delta^{-1} C_1^2 Q)^{2\alpha-2} \|\zeta_{\Omega}\|_2^2.$$

If we have

$$(2.51) \quad \frac{\alpha^3}{\varrho^4} \left(\frac{8}{5}\right)^{2\alpha} \|\psi_{\Theta}\|_2^2 \geq C_8 (1 + K^{\frac{p}{p-1}}) \|\psi_{\Omega}\|_2^2,$$

we obtain

$$(2.52) \quad C_6 \left(\frac{5}{4} C_1\right)^{2\alpha-2} (1 + K^{\frac{p}{p-1}}) \|\psi_{\Omega}\|_2^2 \leq \frac{1}{2} \frac{2\alpha^3}{9C_3 \varrho^4} (2C_1)^{1+2\alpha} \|\psi_{\Theta}\|_2^2,$$

so we conclude that

$$(2.53) \quad \begin{aligned} & \frac{\alpha^3}{9C_3 \varrho^4} (2C_1)^{1+2\alpha} \|\psi_{\Theta}\|_2^2 \\ & \leq C_9 \delta^{-4} (16\delta^{-1} C_1^2 Q)^{2\alpha-2} ((K^{\frac{p}{p-1}} + \delta^{-2}) \|\psi_{0,\delta}\|_2^2 + \|\zeta_{\Omega}\|_2^2). \end{aligned}$$

Thus,

$$(2.54) \quad \frac{\alpha^3}{\varrho^4} Q^4 ((8C_1 Q)^{-1} \delta)^{2\alpha+2} \|\psi_{\Theta}\|_2^2 \leq C_{10} ((K^{\frac{p}{p-1}} + \delta^{-2}) \|\psi_{0,\delta}\|_2^2 + \|\zeta_{\Omega}\|_2^2).$$

Since $(\frac{\delta}{Q})^5 \leq (\frac{1}{2})^5 \leq \frac{1}{8C_1}$ by (1.8), we have

$$(2.55) \quad \frac{\alpha^3}{\varrho^4} Q^6 \left(\frac{\delta}{Q}\right)^{12\alpha+14} \|\psi_{\Theta}\|_2^2 \leq C_{11} ((1 + K^{\frac{p}{p-1}}) \|\psi_{0,\delta}\|_2^2 + \delta^2 \|\zeta_{\Omega}\|_2^2).$$

To satisfy (2.40) and (2.51), we choose

$$(2.56) \quad \alpha = C_{12} (1 + K^{\frac{2p}{3p-4}}) \left(Q^{\frac{4p-4}{3p-4}} + \log \frac{\|\psi_{\Omega}\|_2}{\|\psi_{\Theta}\|_2} \right),$$

Combining with (2.55), and recalling $Q \geq 1$, we get

$$(2.57) \quad \begin{aligned} & (1 + K^{\frac{2p}{3p-4}})^3 \left(\frac{\delta}{Q}\right)^{C_{13}(1+K^{\frac{2p}{3p-4}}) \left(Q^{\frac{4p-4}{3p-4}} + \log \frac{\|\psi_{\Omega}\|_2}{\|\psi_{\Theta}\|_2}\right)} \|\psi_{\Theta}\|_2^2 \\ & \leq C_{14} ((1 + K^{\frac{p}{p-1}}) \|\psi_{0,\delta}\|_2^2 + \delta^2 \|\zeta_{\Omega}\|_2^2), \end{aligned}$$

and hence there exists $m > 0$ such that

$$(2.58) \quad \left(\frac{\delta}{Q}\right)^{m(1+K^{\frac{2p}{3p-4}}) \left(Q^{\frac{4p-4}{3p-4}} + \log \frac{\|\psi_{\Omega}\|_2}{\|\psi_{\Theta}\|_2}\right)} \|\psi_{\Theta}\|_2^2 \leq \|\psi_{x_0,\delta}\|_2^2 + \delta^2 \|\zeta_{\Omega}\|_2^2.$$

If $\|V^{(2)}\|_p \leq K_2 < \infty$ for some $p > 2$, we have $(\| |V^{(2)}|^{p'} \|_{\varphi^*})^{\frac{1}{p'}} \leq K_2$ for any $p' \in [2, p)$ since

$$(2.59) \quad \int_{\mathbb{R}^2} \varphi^* \left(\frac{|V^{(2)}|^{p'}}{K_2^{p'}} \right) dx \leq \int_{\mathbb{R}^2} \left(\frac{|V^{(2)}|^{p'}}{K_2^{p'}} \right)^{\frac{p}{p'}} dx \leq \int_{\mathbb{R}^2} \frac{|V^{(2)}|^p}{K_2^p} dx \leq 1.$$

We conclude that (2.58) holds with p' substituted for p . Letting $p' \uparrow p$ we obtain (2.58) since K_2 is independent of p' .

(c) $d = 1$: Let $\|V^{(2)}\|_p \leq K_2$ with $p \geq 2$. Using Hölder's inequality and (2.8) with $q = 2$, we get

$$(2.60) \quad \int_{\mathbb{R}} W_M^2 \omega_{\varrho}^{2-2\alpha} \eta^2 \psi^2 dx \leq \|W_M\|_2^2 \|\omega_{\varrho}^{2-2\alpha} \eta^2 \psi^2\|_{\infty} \leq M^{-\frac{p-2}{2}} K_2^p \|\omega_{\varrho}^{2-2\alpha} \eta^2 \psi^2\|_{\infty}.$$

Applying Sobolev's inequality, we obtain

$$(2.61) \quad \begin{aligned} \|\omega_{\varrho}^{2-2\alpha} \eta^2 \psi^2\|_{\infty} &\leq \int_{\mathbb{R}} |\omega_{\varrho}^{1-\alpha} \eta \psi|^2 dx + \int_{\mathbb{R}} |(\omega_{\varrho}^{1-\alpha} \eta \psi)'|^2 dx \\ &\leq \int_{\mathbb{R}} |\omega_{\varrho}^{1-\alpha} \eta \psi|^2 dx + 2 \int_{\mathbb{R}} |(\omega_{\varrho}^{1-\alpha})'|^2 \eta^2 \psi^2 dx + 2 \int_{\mathbb{R}} \omega_{\varrho}^{1-2\alpha} |(\eta \psi)'|^2 dx. \end{aligned}$$

Combining (2.7), (2.60), (2.61), and (2.12) with $d = 1$, we conclude that

$$(2.62) \quad \begin{aligned} &\left(\frac{\alpha^3}{3C_3\varrho^4} - 4K_1^2 - 4M - 4M^{-\frac{p-2}{2}} K_2^p - 8C_4 M^{-\frac{p-2}{2}} K_2^p \frac{\alpha^2}{\varrho^2} \right) \int_{\mathbb{R}} \omega_{\varrho}^{-1-2\alpha} \eta^2 \psi^2 dx \\ &\quad + \left(\frac{\alpha}{3C_3\varrho^2} - 8M^{-\frac{p-2}{2}} K_2^p \right) \int_{\mathbb{R}} \omega_{\varrho}^{1-2\alpha} |(\eta \psi)'|^2 dx \\ &\leq 4 \int_{\text{supp } \eta'} \omega_{\varrho}^{2-2\alpha} |\eta'|^2 |\psi'|^2 dx + \int_{\text{supp } \eta'} \omega_{\varrho}^{2-2\alpha} (\eta'')^2 \psi^2 dx \\ &\quad + 2 \int_{\text{supp } \eta} \omega_{\varrho}^{2-2\alpha} \eta^2 \zeta^2 dx. \end{aligned}$$

Assuming $\alpha \geq \varrho$, and setting $M = K_2^2 \alpha^{\frac{4}{p}} \varrho^{-\frac{4}{p}}$, we have

$$(2.63) \quad \begin{aligned} &4K_1^2 + 4M + 4M^{-\frac{p-2}{2}} K_2^p + 8M^{-\frac{p-2}{2}} K_2^p \frac{\alpha^2}{\varrho^2} \\ &\leq 4K_1^2 + 4M + 12M^{-\frac{p-2}{2}} K_2^p \frac{\alpha^2}{\varrho^2} = 4K_1^2 + 16K_2^2 \alpha^{\frac{4}{p}} \varrho^{-\frac{4}{p}} \leq 16K^2 \alpha^{\frac{4}{p}} \varrho^{-\frac{4}{p}}. \end{aligned}$$

Taking

$$(2.64) \quad \alpha \geq C_5 (1 + K^{\frac{2p}{3p-4}}) \varrho^{\frac{4p-4}{3p-4}} \geq C_5 (1 + K^{\frac{2p}{3p-4}}) \varrho^{\frac{4}{3}},$$

we can guarantee that $\alpha > C_2$,

$$(2.65) \quad \frac{\alpha^3}{3C_3\varrho^4} \geq 3(16K^2 \alpha^{\frac{4}{p}} \varrho^{-\frac{4}{p}}),$$

and

$$(2.66) \quad \frac{\alpha}{3C_3\varrho^2} - 8M^{-\frac{p-2}{2}} K_2^p \geq 0.$$

Using (2.1) and recalling (1.6), we obtain

$$(2.67) \quad \int_{\mathbb{R}} \omega_{\varrho}^{-1-2\alpha} \eta^2 \psi^2 dx \geq \left(\frac{\varrho}{Q} \right)^{1+2\alpha} \|\psi_{\Theta}\|_2^2 \geq (2C_1)^{1+2\alpha} \|\psi_{\Theta}\|_2^2.$$

Combining (2.62), (2.65), (2.66) and (2.67), we conclude that

$$(2.68) \quad \begin{aligned} \frac{2\alpha^3}{9C_3\varrho^4}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2 &\leq 4 \int_{\text{supp } \eta'} \omega_\varrho^{2-2\alpha} |\eta'|^2 |\psi'|^2 dx \\ &+ \int_{\text{supp } \eta'} \omega_\varrho^{2-2\alpha} (\eta'')^2 \psi^2 dx + 2 \int_{\text{supp } \eta} \omega_\varrho^{2-2\alpha} \eta^2 \zeta^2 dx \end{aligned}$$

Let $f \in \mathcal{D}(\nabla)$ and $M > 0$. Using (2.20), Hölder's inequality, (2.8) with $d = 1$, and Sobolev's inequality, we get

$$(2.69) \quad \left| \int_{\mathbb{R}} V f^2 dx \right| \leq (K_1 + M^{\frac{1}{2}} + M^{-\frac{p-1}{2}} K_2^p) \|f\|_2^2 + M^{-\frac{p-1}{2}} K_2^p \|f'\|_2^2.$$

Taking $M = (2K_2^p)^{\frac{2}{p-1}}$, we get

$$(2.70) \quad \left| \int_{\mathbb{R}} V f^2 dx \right| \leq 2(1 + K^{\frac{p}{p-1}}) \|f\|_2^2 + \frac{1}{2} \|f'\|_2^2.$$

We have

$$(2.71) \quad \begin{aligned} &\int_{\{2C_1Q \leq |x| \leq 2C_1Q+1\}} \omega_\varrho^{2-2\alpha} (4|\eta'|^2 |\psi'|^2 + (\eta'')^2 \psi^2) dx \\ &\leq 64 \left(\frac{C_1\varrho}{2C_1Q} \right)^{2\alpha-2} \int_{\{2C_1Q \leq |x| \leq 2C_1Q+1\}} (4|\psi'|^2 + \psi^2) dx \\ &\leq C_6 \left(\frac{5}{4} C_1 \right)^{2\alpha-2} \int_{\{2C_1Q-1 \leq |x| \leq 2C_1Q+2\}} (\zeta^2 + (1 + K^{\frac{p}{p-1}}) \psi^2) dx \\ &\leq C_6 \left(\frac{5}{4} C_1 \right)^{2\alpha-2} (\|\zeta_\Omega\|_2^2 + (1 + K^{\frac{p}{p-1}}) \|\psi_\Omega\|_2^2), \end{aligned}$$

where we used (2.47) and an interior estimate. Similarly,

$$(2.72) \quad \begin{aligned} &\int_{\{\frac{\delta}{4} \leq |x| \leq \frac{3\delta}{4}\}} \omega_\varrho^{2-2\alpha} (4|\eta'|^2 |\psi'|^2 + (\eta'')^2 \psi^2) dx \\ &\leq 1024\delta^{-4} (4\delta^{-1} C_1\varrho)^{2\alpha-2} \int_{\{\frac{\delta}{4} \leq |x| \leq \frac{3\delta}{4}\}} (4|\psi'|^2 + \psi^2) dx \\ &\leq C_7\delta^{-4} (4\delta^{-1} C_1\varrho)^{2\alpha-2} \int_{\{|x| \leq \delta\}} (\zeta^2 + (K^{\frac{p}{p-1}} + \delta^{-2}) \psi^2) dx \\ &\leq C_7\delta^{-4} (16\delta^{-1} C_1^2 Q)^{2\alpha-2} (\|\zeta_\Omega\|_2^2 + (K^{\frac{p}{p-1}} + \delta^{-2}) \|\psi_{0,\delta}\|_2^2). \end{aligned}$$

In addition,

$$(2.73) \quad \int_{\text{supp } \eta} \omega_\varrho^{2-2\alpha} \eta^2 \zeta^2 dx \leq (4\delta^{-1} C_1\varrho)^{2\alpha-2} \|\zeta_\Omega\|_2^2 \leq (16\delta^{-1} C_1^2 Q)^{2\alpha-2} \|\zeta_\Omega\|_2^2.$$

If we have

$$(2.74) \quad \frac{\alpha^3}{\varrho^4} \left(\frac{8}{5} \right)^{2\alpha} \|\psi_\Theta\|_2^2 \geq C_8 (1 + K^{\frac{p}{p-1}}) \|\psi_\Omega\|_2^2,$$

we obtain

$$(2.75) \quad C_6 \left(\frac{5}{4} C_1 \right)^{2\alpha-2} (1 + K^{\frac{p}{p-1}}) \|\psi_\Omega\|_2^2 \leq \frac{1}{2} \frac{2\alpha^3}{9C_3\varrho^4} (2C_1)^{1+2\alpha} \|\psi_\Theta\|_2^2,$$

so we conclude that

$$(2.76) \quad \frac{\alpha^3}{9C_3\varrho^4}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2 \leq C_9\delta^{-4}(16\delta^{-1}C_1^2Q)^{2\alpha-2}((K^{\frac{p}{p-1}} + \delta^{-2})\|\psi_{0,\delta}\|_2^2 + \|\zeta_\Omega\|_2^2).$$

Thus,

$$(2.77) \quad \frac{\alpha^3}{\varrho^4}Q^4((8C_1Q)^{-1}\delta)^{2\alpha+2}\|\psi_\Theta\|_2^2 \leq C_{10}((K^{\frac{p}{p-1}} + \delta^{-2})\|\psi_{0,\delta}\|_2^2 + \|\zeta_\Omega\|_2^2).$$

Since $(\frac{\delta}{Q})^5 \leq (\frac{1}{2})^5 \leq \frac{1}{8C_1}$ by (1.8), we have

$$(2.78) \quad \frac{\alpha^3}{\varrho^4}Q^6\left(\frac{\delta}{Q}\right)^{12\alpha+14}\|\psi_\Theta\|_2^2 \leq C_{11}((1 + K^{\frac{p}{p-1}})\|\psi_{0,\delta}\|_2^2 + \delta^2\|\zeta_\Omega\|_2^2).$$

To satisfy (2.64) and (2.74), we choose

$$(2.79) \quad \alpha = C_{12}(1 + K^{\frac{2p}{3p-4}})\left(Q^{\frac{4p-4}{3p-4}} + \log \frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right),$$

Combining with (2.78), and recalling $Q \geq 1$, we get

$$(2.80) \quad (1 + K^{\frac{2p}{3p-4}})^3\left(\frac{\delta}{Q}\right)^{C_{13}(1+K^{\frac{2p}{3p-4}})\left(Q^{\frac{4p-4}{3p-4}} + \log \frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right)}\|\psi_\Theta\|_2^2 \leq C_{14}((1 + K^{\frac{p}{p-1}})\|\psi_{0,\delta}\|_2^2 + \delta^2\|\zeta_\Omega\|_2^2),$$

and hence there exists $m > 0$ such that

$$(2.81) \quad \left(\frac{\delta}{Q}\right)^{m(1+K^{\frac{2p}{3p-4}})\left(Q^{\frac{4p-4}{3p-4}} + \log \frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right)}\|\psi_\Theta\|_2^2 \leq \|\psi_{x_0,\delta}\|_2^2 + \delta^2\|\zeta_\Omega\|_2^2.$$

□

3. UNIQUE CONTINUATION PRINCIPLE FOR SPECTRAL PROJECTIONS OF SCHRÖDINGER OPERATORS WITH SINGULAR POTENTIALS

The following theorem, a consequence of Theorem 1.1, is an extension of [KIN, Theorem B.4] to Schrödinger operators with singular potentials. Theorem 1.2 follows from Theorem 3.1.

Theorem 3.1. *Let $H = -\Delta + V$ be a Schrödinger operator on $L^2(\mathbb{R}^d)$, where $V = V^{(1)} + V^{(2)}$ with $\|V^{(1)}\|_\infty \leq K_1 < \infty$ and $\|V^{(2)}\|_p \leq K_2 < \infty$ with $p \geq d$ for $d \geq 3$, $p > 2$ for $d = 2$, and $p \geq 2$ for $d = 1$. Set $K = K_1 + K_2$. Fix $\delta \in (0, \frac{1}{2}]$, let $\{y_k\}_{k \in \mathbb{Z}^d}$ be sites in \mathbb{R}^d with $B(y_k, \delta) \subset \Lambda_1(k)$ for all $k \in \mathbb{Z}^d$. There exists a constant $M_d > 0$, such that given a rectangle Λ as in (1.11), where $a \in \mathbb{R}^d$ and $L_j \geq 114\sqrt{d}$ for $j = 1, \dots, d$, and a real-valued $\psi \in \mathcal{D}(H_\Lambda)$, we have*

$$(3.1) \quad \delta^{M_d(1+K^{\beta_{d,p}})}\|\psi_\Lambda\|_2^2 \leq \sum_{k \in \mathbb{Z}^d, \Lambda_1(k) \subset \Lambda} \|\psi_{y_k,\delta}\|_2^2 + \delta^2\|(-\Delta + V)\psi\|_\Lambda^2,$$

where

$$(3.2) \quad \beta_{d,p} = \begin{cases} \frac{2p}{3p-2d} & \text{for } d \geq 2 \\ \frac{2p}{3p-4} & \text{for } d = 1 \end{cases}.$$

Proof of Theorem 3.1. Under the hypotheses of the theorem $V \in L^2_{loc}(\mathbb{R}^d)$, which implies that $\mathcal{D}(\Delta_\Lambda) \cap \{\phi \in L^2(\Lambda) : V\phi \in L^2(\Lambda)\}$ is an operator core for H_Λ , so it suffices to prove the theorem for $\psi \in \mathcal{D}(\Delta_\Lambda)$ with $V\psi \in L^2(\Lambda)$.

Using the notation in the proof of [KLN, Theorem B.4], we have $\|\widehat{V^{(1)}}\|_\infty = \|V^{(1)}\|_\infty \leq K_1$ and $\|\widehat{V^{(2)}}\|_{\Lambda_{Y\tau}(\kappa)} \leq 3^d \|V^{(2)}\|_p \leq 3^d K_2$ for any $\kappa \in \Lambda$, since $\Lambda_{Y\tau}(\kappa) \subset \Lambda_{3L}$ as $Y\tau_j < \frac{L_j}{2}, j = 1, 2, \dots, d$. Using Theorem 1.1 and following the proof of [KLN, Theorem B.4], we prove (3.1). \square

Proof of Theorem 1.2. From (2.22), (2.47) and (2.70), there exists a constant $C_d > 0$ such that for all $f \in \mathcal{D}(\nabla)$

$$(3.3) \quad \left| \int_{\mathbb{R}^d} V f^2 dx \right| \leq \theta \|f\|_2^2 + \frac{1}{2} \|\nabla f\|_2^2$$

where $\theta = C_d(1 + K^{\frac{2p}{2p-d}})$ for $d \geq 2$ and $\theta = C_1(1 + K^{\frac{p}{p-1}})$ for $d = 1$. Therefore $\sigma(H_\Lambda) \subset [-\theta, \infty)$, and hence it suffices to consider $E_0 \geq -\theta$ and $E \in [-\theta, E_0]$. We have $V - E = (V^{(1)} - E) + V^{(2)}$, where

$$(3.4) \quad \|V^{(1)} - E\|_\infty \leq \|V^{(1)}\|_\infty + \max\{E_0, \theta\} \leq K_1 + E_0 + \theta$$

and $\|V^{(2)}\|_p \leq K_2$. Applying Theorem 3.1 and following the proof of [KLN, Theorem B.1], we prove (1.13). \square

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